A Supplement to “Frequency Responses of a Neural Oscillator”

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The transfer function $N(\omega, A)$ of the neural oscillator discussed in the article “Frequency Responses of a Neural Oscillator” is a very complicated function of frequency ω and amplitude A of the sinusoidal input. It is

$$N(\omega, A) = K(r_x) \frac{1 + jT\omega}{1 + (\tau T\omega_n^2 - 1)\frac{K(r_x)}{K_n} - \tau T\omega^2 + j(K_n - K(r_x))Ta\omega}, \quad (1)$$

where function K is

$$K(r) = \begin{cases} 0 & (r < -1) \\ \frac{1}{\pi}\left(\sqrt{1-r^2}r - \cos^{-1}(r)\right) + 1 & (-1 \le r \le 1) \\ 1 & (r > 1). \end{cases}$$

Parameter ω_n represents an approximate frequency of the self-excited oscillation, being

$$\omega_n = \frac{1}{T}\sqrt{\frac{(\tau + T)b}{\tau a} - 1}. \quad (2)$$

Parameter K_n is

$$K_n = \frac{\tau + T}{Ta}. \quad (3)$$

If r_x in (1) were independent of ω and A, the system would be just a simple second-order linear one. Actually, r_x is a very complicated function of ω and A. It is implicitly defined by the following equation:

$$\begin{aligned}\{r_x + (a+b)L(r_x)\}&\left|\frac{1+jT\omega}{1+(\tau T\omega_n^2-1)\frac{K(r_x)}{K_n} - \tau T\omega^2 + j(K_n - K(r_x))Ta\omega}\right| \\ &= 2\left(\frac{c}{A} - \frac{1}{\pi}\right),\end{aligned} \quad (4)$$

where function L is defined by

$$L(r) = \begin{cases} 0 & (r < -1) \\ \frac{1}{\pi}\left(\sqrt{1-r^2} - r\cos^{-1}(r)\right) + r & (-1 \le r \le 1) \\ r & (r > 1). \end{cases} \quad (5)$$

In order for the oscillator to generate a stationary oscillation by itself, the model parameters must satisfy the following inequalities

$$1 + \frac{\tau}{T} < a < b + 1. \tag{6}$$

Below we assume that the parameters are set to satisfy this condition, but if the oscillator is used as an input-output system, the condition is not necessarily required. The non-oscillatory oscillator coupled with a controlled object in a feedback manner can produce an oscillation.

The transfer function, (1) with (4), is extremely complex, but it can considerably be simplified by giving some constraints to the model parameters and the input amplitude A. I think this simplification is very helpful to get a rough picture of the frequency response of the oscillator. The first constraint is $a = b$, which always satisfies the second inequality in (6). As is shown in the article, if $a = b$, we have $\omega_n = \frac{1}{\sqrt{\tau T}}$, leading to

$$N(\omega, A) \quad = K(r_x) \frac{1 + jT\omega}{1 - \tau T\omega^2 + j(K_n - K(r_x))Ta\omega}. \tag{7}$$

The second constraint is $A = \pi c$. It makes (4) extremely simple as

$$r_x + 2aL(r_x) = 0. \tag{8}$$

This implies that r_x is independent of ω and A, depending only upon a (> 0). It is easy to see that r_x is a monotonically decreasing function of a and that $r_x = -1$ for $a \to \infty$. From the first inequality of (6), parameter a must be greater than 1. This fact and (8) lead to $L(r_x) < -\frac{1}{2}r_x$. From this inequality and the middle figure in Fig.1, we find that $-1 < r_x \lesssim -0.3$.

The value πc has a special meaning. As is shown in the article "Frequency Responses...", the input amplitude A must satisfy the following inequalities

$$A_0(\omega) < A < A_1(\omega), \tag{9}$$

where

$$A_0(\omega) = \frac{c}{\frac{1}{2}\frac{\sqrt{T^2\omega^2+1}}{\tau T|\omega^2-\omega_n^2|}\frac{c}{A_n} + \frac{1}{\pi}}$$

and

$$A_1(\omega) = \begin{cases} \infty & \left(\omega \leq \omega_1 \doteq \frac{\sqrt{\frac{\pi^2}{4}-1}}{\tau}\right) \\ \frac{c}{\frac{1}{\pi} - \frac{1}{2}\frac{1}{\sqrt{\tau^2\omega^2+1}}} & (\omega > \omega_1). \end{cases}$$

If $A < A_0(\omega)$, then the output of the oscillator is not completely entrained by the input. If $A > A_1(\omega)$, then the output will vanish. (The latter is a peculiar property of the Matsuoka model.) Functions $A_0(\omega)$ and $A_1(\omega)$ both coverages to πc for $\omega \to \infty$. Namely, $A = \pi c$ is the only value that satisfies (9) for every $0 < \omega < \infty$.

For $-1 < r_x \lesssim -0.3$, $L(r_x) \approx \frac{1}{2} K(r_x)$ holds approximately (see the right figure (the thin curve) in Fig.1 and Appendix B) and hence we have $aK(r_x) \approx 2aL(r_x) = -r_x$. Substituting this into (7), we have

$$N(\omega) = K(r_x) \frac{1 + jT\omega}{1 - \tau T\omega^2 + j(K_n a + r_x) T\omega}. \tag{10}$$

Moreover we rewrite this as a function of the normalized frequency $\Omega = \frac{\omega}{\omega_n}$:

$$N(\Omega) = K(r_x) \frac{1 + j\xi\Omega}{1 - \Omega^2 + j\left(\frac{1}{\xi} + (1 + r_x)\xi\right)\Omega}. \tag{11}$$

where $\xi = \sqrt{\frac{T}{\tau}}$. Thus, we have obtained a very simple form of transfer function, though its shape depends on two parameters $\xi > 0$ and $-1 < r_x \lesssim -0.3$.

Now, we can draw a rough picture of the frequency response of the oscillator by investigating (11). As an example, let us find the condition for the system to have a strong resonance characteristic around $\Omega \approx 1$ (or $\omega \approx \omega_n$). The (relative) gain of $N(\Omega)$ at $\Omega \approx 1$ is

$$\frac{|N(1)|}{|N(0)|} = \frac{\sqrt{1 + \xi^2}}{\frac{1}{\xi} + (1 + r_x)\xi}.$$

This value becomes very large only when ξ is very large and r_x is close to -1. Namely, the strong resonance occurs only if $T \gg \tau$ and $a \gg 1$.