Second Supplement to "Frequency Responses of a Neural Oscillator"

Kiyotoshi Matsuoka

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1 The Oscillator with Linear Input

The system dealt with in "Frequency Responses of a Neural Oscillator" is

$$\begin{aligned}
\tau \frac{d}{dt} x_i(t) + x_i(t) &= c - a y_j(t) - b v_i(t) - g(\pm u(t)) \quad (i, j = 1, 2; i \neq j), &(1)\\
T \frac{d}{dt} v_i(t) + v_i(t) &= y_i(t) = g(x_i(t)), &(2)\\
y(t) &= y_2(t) - y_1(t). &(3)
\end{aligned}$$

An important feature of this input-output system is in the way of input, $-g(\pm u(t))$. Since $y_j(t)$, $v_i(t)$ and $g(\pm u(t))$ are all non-negative, the right-hand side of (1) never exceeds positive constant c, implying that $x_i(t)$ is bounded to the upper by c. This leads to $0 \leq y_i(t) \leq c$ and hence $-c \leq y(t) \leq c$. Thus we find that the output $y(t)$ of the system is bounded however positively/negatively large the input $u(t)$ is.

If the nonlinear inputs $-g(\pm u(t))$ is replaced by linear inputs $\mp u(t)$ as

$$\tau \frac{d}{dt} x_i(t) + x_i(t) = c - a y_j(t) - b v_i(t) \mp u(t), \tag{4}$$

then the boundedness of the output will be lost. This supplement discusses the frequency response in this case.

2 Describing Function

Carrying out the same calculations as those in "Frequency Responses ..." for the present system, we have

$$\begin{aligned}
\tau \frac{d}{dt} \widetilde{x}_i(t) + \widetilde{x}_i(t) + A_x r_x &= c - a\left(K(r_x)\tilde{x}_j(t) + A_x L(r_x)\right)\\
&\quad - b\left(\tilde{v}_i(t) + A_x L(r_x)\right)\\
&\quad \mp u(t), &(5)
\end{aligned}$$

$$T\frac{d}{dt}\tilde{v}_i(t) + \tilde{v}_i(t) = K(r_x)\tilde{x}_i(t), \tag{6}$$

$$y_i = K(r_x)\tilde{x}_i(t) + A_x L(r_x). \tag{7}$$

Defining $x(t) \triangleq \widetilde{x}_2(t) - \widetilde{x}_1(t)$ and $v(t) \triangleq \widetilde{v}_2(t) - \widetilde{v}_1(t)$, we obtain a set of linearized equations that relates input $u(t)$ and output $y(t)$:

$$\tau\frac{d}{dt}x(t) + (1 - aK(r_x))x(t) = -bv(t) + 2u(t), \tag{8}$$

$$T\frac{d}{dt}v(t) + v(t) = K(r_x)x(t), \tag{9}$$

$$y(t) = K(r_x)x(t). \tag{10}$$

Applying the Laplace transform to these equations, we obtain the following transfer functions:

$$G(s, A) = \frac{2(Ts+1)}{\tau Ts^2 + (K_n - K(r_x))Tas + 1 + (\tau T\omega_n^2 - 1)\frac{K(r_x)}{K_n}}, \tag{11}$$

$$N(s, A) = K(r_x)G(s, A). \tag{12}$$

Function $G(s, A)$ represents the transfer function from $u(t)$ to $x(t)$, and function $N(s, A)$ represents that from $u(t)$ to $y(t)$. Substituting s in $G(s, A)$ and $N(s, A)$ by $j\omega$, we have frequency transfer functions or describing functions of the system:

$$G(\omega, A) = \frac{2(jT\omega + 1)}{1 + (\tau T\omega_n^2 - 1)\frac{K(r_x)}{K_n} - \tau T\omega^2 + j(K_n - K(r_x))Ta\omega}, \tag{13}$$

$$N(\omega, A) = K(r_x)G(\omega, A). \tag{14}$$

In the same way as "Frequency Response ..." the following holds:

$$A_x = \frac{1}{2}|G(\omega, A)|A. \tag{15}$$

The obtained describing functions do not look so different from those of the previous model; the only difference is that 2 in the numerator of (13) is missing in the previous model. Actually, however, $r_x = r_x(\omega, A)$ is different between them. Extracting the bias components from (5), we have $A_x r_x = c - aA_x L(r_x) - bA_x L(r_x)$ or

$$A_x\{r_x + (a+b)L(r_x)\} = c. \tag{16}$$

Combining (15) and (16), we obtain $\{r_x + (a+b)L(r_x)\}|G(\omega, A)| = \frac{2c}{A}$ or

$$\frac{\{r_x + (a+b)L(r_x)\}\,|jT\omega + 1|}{\left|1 + (\tau T\omega_n^2 - 1)\,\frac{K(r_x)}{K_n} - \tau T\omega^2 + j\,(K_n - K\,(r_x))\,Ta\omega\right|} = \frac{c}{A}. \tag{17}$$

This equation implicitly defines function $r_x = r_x(\omega, A)$. In the previous model the right-hand side of the corresponding equation was $2\left(\frac{c}{A} - \frac{1}{\pi}\right)$.

3 Dependency of the Frequency Response on the Input Amplitude

If the amplitude of the sinusoidal input is small, the output will be a mixture of an oscillator-originated wave and an input-originated one. As the amplitude increases and exceeds a critical value, the so-called entrainment occurs; the inherent oscillation originated from the oscillator completely disappears. Transfer function $G\,(s, A)$ was derived based on the assumption that the oscillator entrains to the sinusoidal input. However, when the input amplitude is too small and hence $K\,(r_x)$ is large, $G\,(s, A)$ enters an unstable region. The critical amplitude below which the system becomes unstable can be considered the lower bound of the entrainment.

Equation (11) implies that the stable-unstable transition occurs when

$$K\,(r_x) = K_n \;\left(= \frac{\tau + T}{Ta}\right). \tag{18}$$

In this situation, (12) with (11)becomes

$$N(s, A) = \frac{2K_n\,(Ts + 1)}{\tau T(s^2 + \omega_n^2)}. \tag{19}$$

The linear system theory tell us that, without input, this system generates a periodic oscillation of frequency ω_n, which is the inherent frequency of the oscillator. Thus, it can be predicted that the complete entrainment occurs when $K(r_x) < K_n$.

Substituting $K\,(r_x) = K_n$ or $r_x = K^{-1}\,(K_n)$ into (17) and solving it with respect to A, we find the minimum amplitude of the input for entrainment. We denote it by $A_0(\omega)$:

$$A_0(\omega) = \frac{\tau T|\omega^2 - \omega_n^2|}{\sqrt{T^2\omega^2 + 1}} A_n, \tag{20}$$

where

$$A_n \triangleq \frac{c}{K^{-1}\,(K_n) + (a+b)\,L\,(K^{-1}\,(K_n))}. \tag{21}$$

In the previous model the minimum amplitude for entrainment was

$$A_0(\omega) = \frac{c}{\frac{1}{2}\frac{\sqrt{T^2\omega^2+1}}{\tau T|\omega^2-\omega_n^2|}\frac{c}{A_n} + \frac{1}{\pi}}. \tag{22}$$

There is a big difference between (20) and (22). Function (20) is an unbounded function with respect to ω; $A_0(\omega) \to \infty$ for $\omega \to \infty$. In contrast, function (22) is bounded; $A_0(\omega) < \pi c$.

Next let us investigate what will happen when the input amplitude is very large. For $A \to \infty$, (17) reduces to

$$r_x + (a+b)L(r_x) = 0.$$

Thus, we find that the present system with $A \to c$ has essentially the same frequency response as that of the previous system with $A = \pi c$; see "A supplement to 'Frequency Responses of a Neural Oscillator"'. In the previous system, as the input amplitude increases further than πc and reach a certain finite value $A_1(\omega)$, the output totally vanishes, i.e., $N(\omega, A_1(\omega)) = 0$. Such a phenomenon does not occur in the present system.